~~[illegible]~~ 10- -21

Last time: Chain "Roolz" Rule, $\frac{\partial f}{\partial t_i} = \frac{\partial F}{\partial x_1} \cdot \frac{\partial f}{\partial t_2} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x}{\partial t_2} + \dots + \frac{\partial f}{\partial x_N} \cdot \frac{\partial x_n}{\partial t_i}$

Implicit function theorem: Let F be a function w/ $\frac{\partial F}{\partial x} \neq 0$ and $\frac{\partial f}{\partial x_i}$ cts. then on the locus (set of points) of $F(x_1, x_2, x_3 \cdots x_N) = 0$ we have locally $x_N = f(x_1, \dots, x_n)$ and $\frac{\partial f}{\partial x_i} = - \frac{\partial F}{\partial x_i} \Big/ \frac{\partial F}{\partial x_N}$

Pf (IFT Derivative Formula): Apply a partial derivative to F using chain rule:

$$0 = \frac{\partial F}{\partial x_1} \cdot \frac{\partial x_1}{\partial x_i} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_i} \cdot \frac{\partial x_2}{\partial x_i} + \dots + \frac{\partial F}{\partial x_N} \cdot \frac{\partial x_N}{\partial x_i}$$

This caused things to "go away"

For $i \neq k$ & $k = N$ we have that ~~the [illegible]~~ $\frac{\partial x_k}{\partial x_j} = 0$

Thus we obtain:

$$0 = \frac{\partial F}{\partial x_i} \cdot \frac{\partial x_i}{\partial x_i} + \frac{\partial F}{\partial x_N} \cdot \frac{\partial x_N}{\partial x_f}$$

$$= \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial N_N} \cdot \frac{\partial f}{\partial x_i}$$

Solving, we obtain $\frac{\partial f}{\partial x_i} = - \frac{\partial F}{\partial x_i} \Big/ \frac{\partial F}{\partial x_N}$

$\boxed{///}$

Ex: Compute $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ for implicit function $z(x, y)$ given by

$$x^3 + y^3 + z^3 = 2xyz - 5$$

Sol: we want to use IFT. $x^3 + y^3 + z^3 = 2xyz - 5$ iff $x^3 + y^3 + z^3 - 2xyz + 5 = 0$

using $F(x, y, z) = x^3 + y^3 + z^3 - 2xyz + 5$, we see

$\frac{\partial F}{\partial x} = 3x^2 - 2yz$, $\frac{\partial F}{\partial y} = 3y^2 - 2xz$, and $\frac{\partial F}{\partial z} = 3z^2 - 2xy$.

Hence, By IFT: $\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z} = -\frac{3x^2 - 2yz}{3z^2 - 2xy}$ & $\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial z}$

$$= -\frac{3y^2 - 2xz}{3z^2 - 2xy} \quad \boxed{//}$$

What is the Derivative of a multivariat function..?

Gradient AND Optimization

Goal: Optimize Functions of Several Variable By extending tricks From Calc I into multi variables.

Def: The gradient of a Function $f(x_1, x_2, \ldots x_n)$ is

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \ldots \frac{\partial f}{\partial x_n} \right\rangle$$

Note: gradient can Be used to clearly restate many of the theorems "Same stuff" that we've seen

① Chain Rule: $\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \vec{x}}{\partial t_i}$

Why? $\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$

By the ChainRule

$$= \underbrace{\left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle}_{\nabla f} \cdot \underbrace{\left\langle \frac{\partial x_1}{\partial t_i}, \frac{\partial x_2}{\partial t_i}, \ldots, \frac{\partial x_n}{\partial t_i} \right\rangle}_{\frac{\partial \vec{x}}{\partial t_i}}$$

$$= \nabla f \cdot \frac{\partial \vec{x}}{\partial t_i}$$

Claim: Directional Derivatives can also Be expressed using the gradient...

Why?: Recall that the Directional Derivative of f at $\vec{P}$ in the direction of $\vec{u}$ is:

$$D_{\vec{u}} f(\vec{P}) = \lim_{h \to 0^+} \frac{f(\vec{P} + h\vec{u}) - f(\vec{P})}{h}$$

Define $g(h) = f(\vec{P} + h\vec{N})$ and Notice $g(0) = f(\vec{P})$

$\therefore D_{\vec{u}} f(\vec{P}) = \lim_{h \to 0^+} \frac{g(h) - g(0)}{N} = g'(0)$ On the other Hand,

$g'(h) = \frac{D}{Dh}[f(\vec{P} + h\vec{u})] = \frac{D}{Dh}[f(P_1 + hu_{(1)}), (P_2 + hu_2), \dots, (P_N + hu_N)]$

Recognize this as a chain rule for $X_i = P_i + hu_i$: $\cancel{g'(h) = \frac{DF}{DX_i}}$ Derive of

$g'(h) = \nabla f(\vec{P} + h\vec{u}) \cdot \frac{D\vec{X}}{Dh} = \nabla f(\vec{P} + h\vec{u}) \cdot \langle u_1, u_2, \dots, u_n \rangle$

$= \nabla f(\vec{P} + hu) \cdot \vec{u}$ $\therefore$ we have $g'(0) = \nabla f(\vec{P} + 0u) \cdot \vec{u} = \nabla f(\vec{P}) \cdot \vec{u}$

Finally we see $D_{\vec{u}} f(\vec{P}) = \nabla f(\vec{P}) \cdot \vec{u}$

EX: lets compute ~~the~~ $D_{\vec{u}} f(\vec{P})$ for $f(x,y) = 4x\sqrt{y}$ at $\vec{P} = \langle 1, 4 \rangle$ in Direction $\vec{u} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

Sol; we know $D_u f(\vec{P}) = \nabla f(\vec{P}) \cdot \vec{u}$ ~~that~~

$\nabla f(x,y) = \langle 4y^{1/2}, 2xy^{-1/2} \rangle$. $\therefore \nabla f(\vec{P}) = \langle 4 \cdot 2, 2 \cdot 1 \cdot \frac{1}{2} \rangle = \langle 8, 1 \rangle$

$\therefore D_{\vec{u}} F(\vec{P}) = \nabla f(\vec{P}) \cdot \vec{u} = \langle 8, 1 \rangle \cdot \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = -\frac{8}{\sqrt{2}} + \frac{1}{\sqrt{2}} = -\frac{7}{\sqrt{2}}$ ☒

5 Minute Break

Question; In which (Chris wants to do another gradient problem, we will return to this material later)

Ex: Compute ∇f for $f(x,y,z) = \frac{xz}{y+z}$.

Sol: $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$. $\frac{\partial f}{\partial x} = \frac{z}{y+z}$, $\frac{\partial f}{\partial y} = -\frac{xz}{(y+z)^2}$, and ~~$\frac{\partial f}{\partial z} = (y+z)\frac{\partial}{\partial z}[xz]$~~ ~~$- xz\frac{\partial}{\partial z}[y+z]$~~

and $\frac{(y+z\frac{\partial}{\partial z}[xy] - xz\frac{\partial}{\partial z}[y+z]}{(y+z)^2}$

$= \frac{(y+z)x - xz \cdot 1}{(y+z)^2}$ $\therefore = \langle \frac{z}{y+z}, -\frac{xz}{(y+z)^2}, \frac{xy}{(y+z)^2} \rangle$

$= \frac{xy}{(y+z)}$

Ex: How do we optimize the directional derivative?

Think about f at $\vec{p}$ and vary unit vector $\vec{u}$

$D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u} = |\nabla f(\vec{p})||\vec{u}| \cos(\theta)$

Just computed earlier

geo int of dot product

$= |\nabla f(\vec{p})| \cos(\theta)$ $\therefore$ Maximizing $D_u f(\vec{p})$ amounts to maximizing $\cos(\theta)$

$\vec{u}$ unit vector

We know from Calc I $\cos(\theta)$ is maximized at $\cos(0) = 1$ (from 0 to π)

$\therefore$ ① the direction of the gradient maximizes directional derivative.
② the maximum directional derivative of f is $|\nabla f(\vec{p})|$

EX: ~~maximize~~ Compute the direction ~~&~~ of max value of $D_{\vec{u}} f(\vec{p})$

For $f(x,y,z) = \frac{xz}{y+z}$ at $\vec{p} = \langle 1,1,-2\rangle$

Sol: We already computed $\nabla f = \langle \frac{z}{y+z}, \frac{xz}{(y+z)^2}, \frac{xy}{(y+z)^2} \rangle$

$\therefore$ at $\vec{P} = \langle 1,1,-2\rangle$, the Dir. Derivative is maximizes in Direction

$\nabla f(1,1,-2) = \langle \frac{-2}{1-2}, \frac{1(-2)}{(1-2)^2}, \frac{1\cdot 1}{(1-2)^2} \rangle = \langle 2,2,1\rangle$

Furthermore, the max value is $|\nabla f(\vec{p})| = |\langle 2,2,1\rangle| = \sqrt{4+4+1} = 3$ $\boxed{q}$

chris says → (Read notes from Calc I about optimization) "You will need a very good grasp on optimization"

(For monday's class)

Def: A function f has...

① a local maximum value at $\vec{p}$ when $f(\vec{p}) \geq f(\vec{x})$ for all $\vec{x}$ near $\vec{p}$.

② a global maximum ~~point~~ value at $\vec{p}$ when $f(\vec{p}) \geq f(\vec{x})$ for all $\vec{x} \in \mathrm{Dom}(f)$

(we call p the (local/global) maximum point for f).

③ minima (Both local & global) are defined similarly [Just flip inequalities

Recall: $F(x) = x$ has none of these...

Q: How do we guarantee existence of extrema? (maxima or extrema)

↳ where do we look for them?

Def: The critical points, point p, $\in \mathrm{Dom}(f)$ a critical point of f when either $\nabla f(\vec{p})$ does not exist or $\nabla f(\vec{p}) = \vec{0}$

Prop (~~Fermat~~ Fermat's extrema theorem): The local extrema of function f occur only at critical points of f